FYUG Even Semester Exam., 2024

MATHEMATICS

(2nd Semester)

Course No. : MATDSC–152T

(Integral Calculus and Vectors)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following questions : 2×10=20

1. Express $\int_a^b f(x)\,dx$ as the limit of sum.

2. Evaluate :

$$\int_{-5}^{5} |x+2|\,dx$$

3. Prove that

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

4. If $I_n = \int x^n \cos ax\,dx$ and $J_n = \int x^n \sin ax\,dx$, then show that $aI_n = x^n \sin ax - nJ_{n-1}$.

5. Evaluate :

$$\int_0^{\pi/2} \cos^6 x\,dx$$

6. Evaluate :

$$\int_0^1 x^2 (1-x)^{3/2}\,dx$$

7. Write the formula to find the length of the curve in Cartesian form and polar form.

8. The circle $x^2 + y^2 = a^2$ revolves around the axis. Write down the surface area and the volume of the whole surface generated.

9. Find the volume generated by revolving about OX, the area bounded by $y = x^3$ between $x = 0$ and $x = 2$.

10. Prove that

$$\hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k}) = 2\vec{a}$$

11. Find the value of λ if the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - \hat{k}$ and $\lambda\hat{i} - 4\hat{j} + 5\hat{k}$ are coplanar.

12. Find the vector equation of a plane in normal form.

13. Show that

$$\frac{d}{dt}\left(\vec{a} \times \frac{d\vec{b}}{dt} - \frac{d\vec{a}}{dt} \times \vec{b}\right) = \vec{a} \times \frac{d^2\vec{b}}{dt^2} - \frac{d^2\vec{a}}{dt^2} \times \vec{b}$$

14. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, then show that

$$\vec{\nabla} r^n = n r^{n-2} \vec{r}$$

15. Show that the vector

$$\vec{V} = (4xy - z^3)\hat{i} + 2x^2\hat{j} - 3zx^2\hat{k}$$

is irrotational.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. *(a)* Evaluate : 5

$$\underset{n\to\infty}{\text{Lt}} \left[\frac{1^2}{n^3 + 1^3} + \frac{2^2}{n^3 + 2^3} + \cdots + \frac{n^2}{2n^3}\right]$$

(b) Prove that

$$\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2$$ 5

17. (a) Evaluate $\int_0^2 e^x$ as the limit of a sum. 5

(b) Evaluate : 5

$$\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{2}$$

18. (a) If $I_n = \int_0^{\pi/4} \tan^n \theta \, d\theta$, then show that

$$I_n = \frac{1}{n-1} - I_{n-2}$$

Hence find the value of $\int_0^{\pi/4} \tan^6 \theta \, d\theta$.

3+2=5

(b) Obtain the reduction formula for

$$\int \sec^n x \, dx$$ 5

19. (a) If $I_n = \int_0^{\pi/2} x^n \sin x \, dx$ and $n > 1$, then show that

$$I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$ 5

(b) Obtain the reduction formula for

$$\int \sin^m x \cos^n x \, dx$$ 5

20. (a) Find the length of the perimeter of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$. 5

(b) Find the surface area of the solid generated by revolving the cardioid $r = a(1-\cos\theta)$ about the initial line. 5

21. (a) Prove that the length of one arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ is $8a$. 5

(b) Find the volume generated by the revolution about x-axis of the area bounded by the loop of the curve $y^2 = x^2(2-x)$. 5

22. (a) (i) If $\vec{\alpha} = 2\hat{i} + 3\hat{j} - 5\hat{k}$, $\vec{\beta} = -\hat{i} + \hat{j} + \hat{k}$ and $\vec{\gamma} = 4\hat{i} + 2\hat{j} + 6\hat{k}$, then find $(\vec{\alpha} \times \vec{\beta}) \cdot \vec{\gamma}$. 2

(ii) Show that

$$(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{c}$$ 3

(b) (i) Find the vector equation of the line joining the points $\hat{i} - 2\hat{j} + \hat{k}$ and $3\hat{k} - 2\hat{j}$. 2

(ii) Find the vector equation of the sphere with the join of two given points as diameter. 3

23. (a) Show that the equation of the straight line passing through $A(\vec{a})$ and parallel to the unit vector $\hat{e}$ is $(\vec{r}-\vec{a})\times\hat{e}=0$. Also deduce that the equation of the straight line passing through the origin is $\vec{r}\times\hat{e}=0$. 3+2=5

(b) Find the condition that the two spheres $\vec{r}^2-\vec{r}\cdot\vec{c}+k=0$ and $\vec{r}^2-2\vec{r}\cdot\vec{c'}+k'=0$ may intersect orthogonally. 5

24. (a) If $\vec{u}(t)$ and $\vec{v}(t)$ be two differentiable functions of the scalar t, then show that

$$\frac{d}{dt}(\vec{u}\times\vec{v})=\vec{u}\times\frac{d\vec{v}}{dt}+\frac{d\vec{u}}{dt}\times\vec{v}$$

Hence show that $\frac{d}{dt}\left(\vec{u}\times\frac{d\vec{u}}{dt}\right)=\vec{u}\times\frac{d^2\vec{u}}{dt^2}$.

3+2=5

(b) If $\vec{f}$ and $\vec{g}$ be two vector point functions, then prove that

$$\text{grad}(\vec{f}\cdot\vec{g})=\vec{f}\times\text{curl }\vec{g}+\vec{g}\times\text{curl }\vec{f}+(\vec{f}\cdot\vec{\nabla})\vec{g}+(\vec{g}\cdot\vec{\nabla})\vec{f}$$ 5

25. (a) (i) Show that

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{a})=\vec{\nabla}(\vec{\nabla}\cdot\vec{a})-\vec{\nabla}^2\vec{a}$$ 3

(ii) Prove that curl (grad $\vec{f}$) $=\vec{0}$. 2

(b) Show that a necessary and sufficient condition for a vector $\vec{r}=\vec{f}(t)$ to have constant magnitude is

$$\vec{f}\cdot\frac{d\vec{f}}{dt}=0$$ 5

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